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# On the solutions of the $CP^1$ model in $(2+1)$ dimensions

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## Abstract

We use the methods of group theory to reduce the equations of motion of the $CP^1$ model in (2+1) dimensions to sets of two coupled ordinary differential equations. We decouple and solve many of these equations in terms of elementary functions, elliptic functions and Painlevé transcendents. Some of the reduced equations do not have the Painlevé property thus indicating that the model is not integrable, while it still posesses many properties of integrable systems (such as stable "numerical" solitons).

## I. INTRODUCTION

Over the past few years, it has become clear that many physical processes can be described well in terms of various partial nonlinear differential equations. The areas providing such equations, range from solid state physics, hydrodynamics and particle physics to biophysics and biochemistry. As the equations are nonlinear, in general, they are hard to solve; in fact, so far no general method of solving these equations is known and each equation has to be treated on its own. However, a particular class of equations, which are derived from the so-called integrable models can be solved using some very general techniques [1] (inverse

scattering methods, Bdcklund transformations, ... ). These equations are very special and their solutions have very special properties. Many of these equations have solutions which are localised in space and propagate at a constant speed. Such solutions, usually called solitons, or extended structures in general, have received a lot of attention in recent years. However, most of these equations depend on two space (or space-time) variables and, as such, can only describe phenomena which are quasi-twodimensional. When they involve more variables, either all variables come in a very nonsymmetric way or the models are very special.

Most applications in nature involve 3 spatial dimensions, and in many applications all spatial variables come on an equal footing. When the applications involve, for example, particle physics or relativity, the underlying models are Lorentz covariant. Such models are, generally, nonintegrable and the methods mentioned above do not apply. On the other hand, some of them can be studied numerically. Such studies have involved full simulations of similar models in (2+1) dimensions or simulations of various reduced models (i.e. approximations to the original models). Some of these studies [2] have found that even though the models were not integrable the behaviour of their extended structures resembled the behaviour that was expected had the models been integrable (i.e. the structures preserved their shapes well, there was little radiation etc.) Morever, the approximate methods [3] gave results which were virtually indistinguishable from the results of full simulations. Hence we feel that the behaviour of integrable models may not be that very unusual; other models which, strictly speaking, are not integrable maybe be "almost" so; with their extended structures showing in their behaviour very little difference from what may be expected had the models been integrable. Moreover, various approximate methods work well and may be used to provide some insight to the behaviour of the solutions of the full equations.

Most of the results, which involve models in (2+1) dimensions, were obtained in the so-called $CP^1$ model (also called the $S^2$ or $O(3)$ model) and its modifications. [2] This model, in its original version [4], is probably the simplest model in (2+1) dimensions which is relativistically covariant and which admits the existence of localised soliton-like solutions.

The $CP^1\,\sigma$ model is defined by the Lagrangian density

$$\mathcal{L}_\sigma = \frac{\infty}{\triangle}(\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}), \tag{1.1}$$

together with the constraint $\vec{\phi}\,.\,\vec{\phi} = 1$. The Euler-Lagrange equations derived from (1.1) are

$$\partial^\mu \partial_\mu \vec{\phi} + (\partial^\mu \vec{\phi} \cdot \partial_\mu \vec{\phi})\vec{\phi} = \vec{0}. \tag{1.2}$$

The model can be modified by the addition of further terms to the Lagrangian density. The terms that have been studied the most extensively involve the (2+1) dimensional analogue of the "Skyrme" term [5] and various "potential" terms [2,6]. They were added, primarily, to stabilise the soliton-like structures. In the original model the soliton-like structures were not really stable; any perturbation would induce their shrinking, or expanding which they could do without any cost of energy due to the conformal invariance of the pure $CP^1$ model. Apart from curing the shrinking, and inducing also weak forces between soliton-like structures the additional forces had little effect on the dynamics of these structures. Moreover, the affects of nonintegrability of these models, were also not that different from similar effects in the pure $CP^1$ case. Hence the dynamics of these models was described well by the dynamics of the $CP^1$ case. The same was true when one looked at the approximate methods [7].

These observations suggest that a lot can be learnt from looking at exact solutions of the $CP^1$ model using the group theoretical method of symmetry reduction [8–10]. This method exploits the symmetry of the original equations to find solutions invariant under some subgroup of the symmetry group (the classic example one can give here involves seeking solutions in two dimensions which are rotationally invariant). The method puts all such attempts on a unified footing and it has been applied with success to many equations. The method leads to equations whose solutions represent specific solutions of the full equations; the solutions are determined locally and the method does not tell us whether these solutions are stable or not with respect to any perturbations.

In our case we would like to apply this method to looking for solutions of the original $CP^1$ model; from the remarks made above we can hope that these solutions will be also

approximate solutions of the modified models. Their stability is harder to predict; but again, guided by the experience from the numerical simulations we hope that, at least, some of them will be stable with respect to small perturbations.

In order to perform the symmetry reductions of the pure $CP^1$ model in (2+1) dimensions we have to decide what variable to use. To avoid having to use the constrained variables ($\vec{\phi}$) it is convenient to use the $W$ formulation of the model which involved the stereographic projection of the sphere $\vec{\phi}\cdot\vec{\phi}=1$ onto the complex plane. In this formulation instead of using the $\phi$ fields we express all the dependence on $\phi$ in terms of their stereographic projection onto the complex plane $W$. The $\phi$ fields are then related to $W$ by

$$\phi^1 = \frac{W+W^*}{1+|W|^2}, \quad \phi^2 = i\frac{W-W^*}{1+|W|^2}, \quad \phi^3 = \frac{1-|W|^2}{1+|W|^2}. \tag{1.3}$$

In this formulation the Lagrangian density becomes

$$L = \frac{\partial_\mu W \partial^\mu W^*}{(1+|W|^2)^2}, \tag{1.4}$$

where $^*$ denotes complex conjugation.

To perform our analysis it is convenient to use the polar version of the $W$ variables; i.e. to put $W = R\exp(i\psi)$ and then study the equations for $R$ and $\psi$. The advantage of this approach is that the equations become simple; the disadvantage comes from having to pay attention that $R$ is real and $\psi$ should be periodic with a period of $2\pi$. (If the period is not $2\pi$ then the solution may become multi-valued etc.) Thus if we find solutions that do not obey these restrictions, then these solutions, however interesting they may be, cannot in general be treated as solutions of the original model.

The equations for $R$ and $\psi$ take the form

$$\partial_{tt}\psi - \partial_{xx}\psi - \partial_{yy}\psi + 2\frac{(1-R^2)}{R(1+R^2)}(\partial_t\psi\partial_t R - \partial_x\psi\partial_x R - \partial_y\psi\partial_y R) = 0 \tag{1.5}$$

and

$$\begin{aligned}\partial_{tt}R - \partial_{xx}R - \partial_{yy}R - \frac{R(1-R^2)}{(1+R^2)}(\partial_t\psi\partial_t\psi - \partial_x\psi\partial_x\psi - \partial_y\psi\partial_y\psi) \\ - \frac{2R}{(1+R^2)}\Big((\partial_t R)^2 - (\partial_x R)^2 - (\partial_y R)^2\Big) = 0.\end{aligned} \tag{1.6}$$

Note, that if we put $R = 1$ the second equation is automatically satisfied and the first one reduces to

$$\partial_{tt}\psi - \partial_{xx}\psi - \partial_{yy}\psi = 0 \tag{1.7}$$

i.e. the linear wave equation for the phase $\psi$.

In section II we determine the symmetry group of equations (1.5) and (1.6). In the following section we present coupled pairs of reduced ordinary differential equations (ODE's) for all two-dimensional subgroups of the symmetry group. Sections IV and V are devoted to the presentation of explicit solutions. We finish the paper with a short discussion of the derived solutions, their relation to the solutions known before and their physical relevance.

## II. THE SYMMETRY GROUP AND ITS SUBGROUPS

The symmetry group of the system (1.5) and (1.6) can be calculated using standard methods [8–10]. We actually made use of a MACSYMA package [11] that provides a simplified and partially solved set of determining equations.

Solving those we find that the symmetry group has the structure of a direct product, namely,

$$G \sim \mathrm{SIM}(2,1) \otimes SU(2), \tag{2.1}$$

where $\mathrm{SIM}(2,1)$ is the similitude group of (2+1) dimensional Minkowski space (the Poincaré group extended by dilations). The group $SU(2)$ rotates the components of the fields amongst each other.

The corresponding Lie algebras $\mathrm{sim}(2,1)$ and $su(2)$ can be represented by vector fields acting on $R$ and $\psi$ and the space-time coordinates. A suitable basis is given by two Lorentz boosts $K_1$ and $K_2$, one rotation $L$, three translations $P_0$, $P_1$ and $P_2$, one dilation $D$ and

three $su(2)$ generators $X$, $Y$ and $Z$. We have

$$
\begin{aligned}
K_1 = -(x\partial_t + t\partial_x), \quad K_2 = -(y\partial_t + t\partial_y), \quad L = -x\partial_y + y\partial_x \\
P_0 = \partial_t, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \quad D = t\partial_t + x\partial_x + y\partial_y,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
X &= \frac{1}{2}\Big(\Big(R - \frac{1}{R}\Big)\sin\psi\partial_\psi + \Big(R^2+1\Big)\cos\psi\partial_R\Big), \\
Y &= \frac{1}{2}\Big(\Big(R - \frac{1}{R}\Big)\cos\psi\partial_\psi - \Big(R^2+1\Big)\sin\psi\partial_R\Big), \\
Z &= \partial_\psi.
\end{aligned}
\tag{2.3}
$$

Our aim is to obtain solutions of eq. (1.5) and (1.6 by the method of symmetry reduction [8–10]. In practice we shall require that solutions are invariant under a two-dimensional subgroup of the symmetry group $G$. This will reduce the original partial differential equations (1.5), (1.6) to a system of ODEs. Subalgebras of $\mathrm{sim}(2,1)$ were classified in ref. [12]. A two-dimensional algebra $\{\hat{A}, \hat{B}\}$ can be either Abelian, $[\hat{A}, \hat{B}] = 0$, or solvable non-Abelian, $[\hat{A}, \hat{B}] = \hat{A}$. Subalgebras of the direct sum $\mathrm{sim}(2,1) \oplus su(2)$ can be obtained by the Goursat "twist" method. [13,14]

The result is the following. There exist 10 parametric classes of Abelian subalgebras, represented by

$$
\{\hat{A} + aZ, \quad \hat{B} + bZ\}
\tag{2.4}
$$

with $a, b \in \mathbb{R}$, $Z$ as in (2.3) and $\{\hat{A}, \hat{B}\}$ equal to one of the following pairs:

$$
\begin{gathered}
\{K_1, P_2\}; \quad \{D, K_2 + L\}; \quad \{K_2 + L, P_0 - P_1\}; \quad \{P_2, P_0 - P_1\}; \quad \{L, P_0\}; \\
\{P_1, P_2\}; \quad \{P_0, P_1\}; \quad \{D, K_1\}; \quad \{D, L\}; \quad \{D - K_1, P_0 - P_1\}.
\end{gathered}
\tag{2.5}
$$

Further, there exist 15 parametric classes of nonabelian two-dimensional subalgebras represented by

$$
\hat{A} + cZ, \hat{B} \qquad c \in \mathbb{R},
\tag{2.6}
$$

where $\{\hat{A}, \hat{B}\}$ is one of the pairs:

$$
\begin{gathered}
\{K_1, K_2 + L\}; \{D, P_2\}; \{K_1, P_0 - P_1\}; \{D, P_0 - P_1\}; \{K_1 + \epsilon P_2, P_0 - P_1\}; \\
\{D + \epsilon(K_2 + L), P_0 - P_1\}; \{D, P_0\}; \{D + aK_1, K_2 + L\}; \{D + aK_1, P_2\}; \\
\{D - K_1 + \epsilon(P_0 - P_1), K_2 + L\}; \{D + K_2 + \epsilon(P_0 + P_2), P_1\}; \{D + aL, P_0\}; \\
\{D + aK_1, P_0 - P_1\}; \{D + K_1 + \epsilon(P_0 + P_1), P_0 - P_1\}; \\
\{D + \frac{1}{2}K_1, K_2 + L + \epsilon(P_0 + P_1)\}
\end{gathered} \tag{2.7}
$$

with $a \in \mathbb{R}$ and $\epsilon = \pm 1$.

## III. THE REDUCED EQUATIONS

For each subgroup (2.4) and (2.6) we find three invariants, $\xi$, $R$ and $F$, using standard methods. [8–10] In terms of these we express the two functions $R$ and $\psi$ of (1.5) and (1.6) as

$$R = R(\xi), \quad \psi = \alpha(x, y, t) + F(\xi), \quad \xi = \xi(x, y, t), \tag{3.1}$$

where $\alpha$ and $\xi$ are given for each subalgebra in Table I and II. Derivatives with respect to the variable $\xi$ will be denoted by dots. We introduce the two invariant operators $\Delta$ and $\nabla^2$ by setting

$$\Delta f = f_{tt} - f_{xx} - f_{yy}, \quad (\nabla f, \nabla g) = f_t g_t - f_x g_x - f_y g_y \tag{3.2}$$

and consider three cases separately.

1. $(\nabla\xi)^2 \neq 0$. We put

$$\frac{\Delta\xi}{(\nabla\xi)^2} = -p = -\frac{\dot{g}}{g}, \quad \frac{(\nabla\alpha, \nabla\xi)}{(\nabla\xi)^2} = h, \quad \frac{\Delta\alpha}{(\nabla\xi)^2} = s, \quad \frac{(\nabla\alpha)^2}{(\nabla\xi)^2} = l, \tag{3.3}$$

with $g = g(\xi)$, $h = h(\xi)$, $s = s(\xi)$ and $l = l(\xi)$.

For further use we also introduce

$$m = \dot{h} - \frac{\dot{g}}{g}h - s \tag{3.4}$$

and

$$B = 2\frac{h^2 - l}{g^2}. \tag{3.5}$$

The two PDE's (1.5) and (1.6) now reduce to

$$\ddot{F} + 2\dot{R}\dot{F}\frac{(1-R^2)}{R(1+R^2)} - \frac{\dot{g}}{g}\dot{F} + 2\frac{(1-R^2)}{R(1+R^2)}\dot{R}h + s = 0 \tag{3.6}$$

$$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 - \frac{R(1-R^2)}{1+R^2}\dot{F}^2 - \frac{\dot{g}}{g}\dot{R} - 2\frac{R(1-R^2)}{1+R^2}\dot{F}h - \frac{R(1-R^2)}{1+R^2}l = 0 \tag{3.7}$$

For each algebra the functions (3.3) are given in Table I.

In order to solve the above system we must decouple its two equations. Putting $\dot{F} + h = V$ we first rewrite equation (3.6) as

$$\dot{V} + 2\dot{R}\frac{(1-R^2)}{R(1+R^2)}V - \frac{\dot{g}}{g}V - m = 0 \tag{3.8}$$

with $m$ as in eq. (3.4).

For $m = 0$ we solve (3.6) and obtain

$$\dot{F} = A\frac{(1+R^2)^2}{R^2}g(\xi) - h, \qquad A = \text{const.} \tag{3.9}$$

Next we substitute $\dot{F}$ into equation (3.7) and obtain a second order ODE for $R(\xi)$

$$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 - \frac{\dot{g}}{g}\dot{R} - A^2g^2\frac{(1-R^2)(1+R^2)^3}{R^3} + \frac{R(1-R^2)}{1+R^2}(h^2 - l) = 0. \tag{3.10}$$

For $m \neq 0$ eq. (3.8) is inhomogeneous. We can still decouple it by putting

$$\dot{F} = \frac{Um}{\dot{U}} - h \tag{3.11}$$

$$\dot{U} = \frac{mR^2}{(R^2+1)^2 g}. \tag{3.12}$$

Using eq. (3.11) we can rewrite (3.7) as a third order ODE for the auxiliary function $U(\xi)$. If we can solve it we obtain $R(\xi)$ from (3.12). However, in this paper we restrict our attention to the case of $m(\xi) = 0$.

2. $(\nabla\xi)^2 = 0$, $\Delta\xi = 0$, $(\nabla\alpha, \nabla\xi) \neq 0$. The reduced equations decouple immediately and we have

$$
\begin{aligned}
(\nabla\alpha, \nabla\xi)\frac{(1-R^2)}{R(1+R^2)}\dot{R} &= -\frac{1}{2}\Delta\alpha, \\
(\nabla\alpha, \nabla\xi)\dot{F} &= -\frac{1}{2}(\nabla\alpha)^2.
\end{aligned}
\tag{3.13}
$$

For $(\nabla\alpha, \nabla\xi) \neq 0$ we can integrate directly to obtain

$$
\frac{R}{R^2+1} = C\exp\left\{-\frac{1}{2}\int\frac{\Delta\alpha}{(\nabla\alpha, \nabla\xi)}\,d\xi\right\}
\tag{3.14}
$$

$$
F = -\frac{1}{2}\int\frac{(\nabla\alpha)^2}{(\nabla\alpha, \nabla\xi)}\,d\xi + F_0.
\tag{3.15}
$$

For $(\nabla\alpha, \nabla\xi) = 0$ we must have also $\Delta\alpha = 0$, $(\nabla\xi)^2 = 0$. Then $R(\xi)$ and $F(\xi)$ are arbitrary functions of $\xi$. In particular this is true for $\alpha = 0$.

3. $(\nabla\xi)^2 = 0$, $\Delta\xi = h(\xi) \neq 0$. This case can actually not occur, since these two equations are compatible only for $h(\xi) = 0$.

## IV. ANALYSIS OF SECOND ORDER ODE

### A. General Comments

In order to obtain explicit solutions we need to solve the ODE (3.10) for the function $R(\xi)$. This equation is in the class analysed by Painlevé and Gambier [15–17]; namely it is of the form

$$
\ddot{y} = f(\dot{y}, y, x)
\tag{4.1}
$$

where $f$ is rational in $\dot{y}$ and $y$ and analytical in $x$. If this equation has the Painlevé property (no movable singularities other than poles) then it can be transformed into one of the 50 standard equations listed e.g. by Ince [17]. The Painlevé test [18,19] provides us with necessary (but not sufficient) conditions for eq. (4.1) to have the Painlevé property. The solution is expanded about an arbitrary point $x_0$ in the complex $x$-plane in a Laurent series

$$
y(x) = \sum_{k=0}^{\infty} a_k \tau^{k+p}, \quad \tau = x - x_0,
\tag{4.2}
$$

where $p$ is required to be an integer (usually a negative one). The coefficients are obtained from a recursion relation of the form

$$P_k a_k = h_k(x_0, a_0, a_1, \ldots a_{k-1}). \tag{4.3}$$

Since (4.2) is supposed to represent a general solution it must depend on two constants; $x_0$ and $a_r$ for some nonnegative $r$, called a resonance value. This occurs if the function $P_r$ satifies $P_r = 0$. Then $a_r$ is arbitrary and we have a consistency condition, the "resonance condition", $h_r(x_0, a_0, a_1, \ldots a_{r-1}) = 0$ (which must be satisfied identically in $x_0$). If the above conditions are satisfied, the Painlevé test is passed and (4.2) represents a two parameter family of formal solutions (locally, within the radius of convergence of the series).

Turning our attention to eq. (3.10) we note that the cases $A \neq 0$ and $A = 0$ must be treated separately.

### B. The case $A \neq 0$

The Painlevé test applied directly fails immediately since a balance between the most singular terms occurs for $p = -\frac{1}{2}$, i.e. we have a movable square root branch point. To remedy this problem we put

$$R(\xi) = \sqrt{-U(\xi)}, \tag{4.4}$$

and obtain

$$\ddot{U} = \dot{U}^2 \left[ \frac{1}{2U} + \frac{1}{U-1} \right] + 2A^2 g^2 \frac{(1+U)(1-U)^3}{U} + \dot{U}\frac{\dot{g}}{g} + 2(h^2 - l)\frac{U(1+U)}{(U-1)}. \tag{4.5}$$

We can now choose a new variable $\eta$ to be

$$\eta = \int g(\xi)\, d\xi \tag{4.6}$$

and transform eq. (4.5) into

$$\ddot{U} = \dot{U}^2 \left[ \frac{1}{2U} + \frac{1}{U-1} \right] + 2A^2 \frac{(1+U)(1-U)^3}{U} + B\frac{U(1+U)}{(U-1)} \tag{4.7}$$

with $B$ as in (3.5).

For $B$=const this is eq. PXXXVIII listed e.g. by Ince [17]. It has a first integral $K$ that we use to write a first order equation for $U$:

$$\dot{U}^2 = -4A^2U^4 + 4KU^3 + (8A^2 - 2B - 8K)U^2 + 4KU - 4A^2. \tag{4.8}$$

Since we have $A \neq 0$ we can rewrite (4.8) as

$$(\dot{U})^2 = -4A^2(U - U_1)(U - U_2)(U - U_3)(U - U_4), \tag{4.9}$$

where the constant roots of the right hand side of (4.9) $U_i$, $i = 1 \dots 4$ satisfy

$$\begin{aligned} U_1U_2U_3U_4 &= 1 \\ U_1U_2U_3 + U_1U_2U_4 + U_1U_3U_4 + U_2U_3U_4 &= \frac{K}{A^2} \\ U_1U_2 + U_1U_3 + U_1U_4 + U_2U_3 + U_2U_4 + U_3U_4 &= -2 + \frac{B}{2A^2} + \frac{2K}{A^2} \\ U_1 + U_2 + U_3 + U_4 &= \frac{K}{A^2}. \end{aligned} \tag{4.10}$$

Eq. (4.9) has elementary algebraic and trigonometric solutions, as well as solutions which resemble solitary waves or kink-like structures (in the symmetry variable $\eta$) in the case of multiple roots. If all $U_i$ are distinct then the solutions of (4.8) involve elliptic functions [20]. Explicit solutions will be presented in the next section.

If $B$ in eq. (4.8) is not constant we proceed differently. We again introduce a new independent variable

$$\eta = e^{\int g \, d\xi} \tag{4.11}$$

and transform eq. (4.5) into

$$\ddot{U} = \dot{U}^2\left[\frac{1}{2U} + \frac{1}{U-1}\right] - \frac{\dot{U}}{\eta} + \frac{2A^2}{\eta^2}(1-U)^2(\frac{1}{U} - U) + 2\frac{h^2 - l}{g^2\eta^2}\frac{U(U+1)}{U-1}. \tag{4.12}$$

For

$$2\frac{h^2 - l}{g^2\eta^2} = \delta = \text{const} \tag{4.13}$$

this is the equation for the fifth Painlevé transcendent.

The values $m = 0$ and $B \neq$ const occur for the cases 1 and 2 from Table I and we actually have

$$\eta = \xi, \qquad \delta = 2\Big(b^2 \pm \frac{a^2}{\eta^2}\Big), \tag{4.14}$$

so $\delta$ =const requires $a = 0$.

Hence we obtain the solutions

$$\begin{gathered} U(\xi) = P_V(\alpha, \beta, \gamma, \delta; \xi) \\ \alpha = -\beta = 2A^2, \quad \gamma = 0, \quad \delta = 2b^2, \end{gathered} \tag{4.15}$$

for equations describing the group reductions 1 and 2 (of Table I) in the case when $a = 0$.

For $b = 0$ in the cases of these two reductions we have $B = 2a^2 =$ const and we obtain solutions in terms of equation (4.8).

### C. The case $A = 0$, $B =$ const

The transformation (4.4) can also be performed for the case $A = 0$. We use the first integral to write our equation as

$$\begin{gathered} \dot{U}^2 = 4K(U - U_1)(U - U_2) \\ U_{1,2} = L + 1 \pm \sqrt{L(L+2)}, \quad L = \frac{B}{4K}, \quad K \neq 0. \end{gathered} \tag{4.16}$$

For $K = 0$ we find a solution immediately; namely we have

$$U = U_0 e^{\pm\sqrt{-2B}\eta}, \quad B \leq 0. \tag{4.17}$$

On the other hand, for $A = 0$ in eq. (3.10), the Painlevé expansion (4.2) gives us $p = -1$ for the leading (most singular) power. Hence the transformation (4.4) is not required and so we can transform eq. (4.2) directly into one of the standard forms.

We put

$$R = -i\frac{Z(\eta) + \mu(\xi)}{Z(\eta) - \mu(\xi)}, \quad \eta = \eta(\xi), \quad \mu \neq 0, \quad \dot{\eta} \neq 0 \tag{4.18}$$

and obtain

$$\ddot{Z} = \frac{\dot{Z}^2}{Z} - \frac{1}{\dot{\eta}}\Big(\frac{\ddot{\eta}}{\dot{\eta}} - \frac{\dot{g}}{g}\Big)\dot{Z} + \frac{1}{\dot{\eta}^2\mu}\Big(\ddot{\mu} - \frac{\dot{\mu}^2}{\mu} - \frac{\dot{g}}{g}\dot{\mu}\Big)Z + \frac{h^2 - l}{4\dot{\eta}^2\mu^2}\frac{Z^4 - \mu^4}{Z}. \tag{4.19}$$

If $B$ of eq. (3.5) is constant we can choose $\eta$ as in eq. (4.6), set $\mu = 1$ and obtain the equation

$$\ddot{Z} = \frac{\dot{Z}^2}{Z} + \frac{B}{8}\Big(\frac{Z^4 - 1}{Z}\Big). \tag{4.20}$$

This equation can be integrated directly for $B = 0$. For $B \neq 0$ it has a first integral $K$ in terms of which we obtain a first order ODE

$$\begin{gathered} \dot{Z}^2 = \frac{B}{8}(Z^2 - Z_1^2)(Z^2 - Z_2^2) \\ Z_{1,2}^2 = -K \pm \sqrt{K^2 - 1}. \end{gathered} \tag{4.21}$$

We again have elliptic function solutions. They are however not new. Since $R$ is real (and nonnegative) we must require that

$$Z = e^{i\sigma(\eta)}, \quad \text{for } \mu = 1, 0 \leq \eta < 2\pi. \tag{4.22}$$

The relation between the function $\sigma(\eta)$ and $U(\eta)$ of eq. (4.1) is

$$\sqrt{-U(\eta)} = \Big|\frac{\sin\sigma}{1 - \sin\sigma}\Big|. \tag{4.23}$$

For $B \neq$const we use the variable (4.10), again set $\mu = 1$ and reduce eq. (4.19) to

$$\ddot{Z} = \frac{\dot{Z}^2}{Z} - \frac{1}{\eta}\dot{Z} + \frac{h^2 - l}{4g^2\eta^2}\frac{Z^4 - 1}{Z}. \tag{4.24}$$

For $\delta$, defined by (4.13), being constant ($\delta = \delta_0$) eq. (4.24) is a special case of the third Painlevé transcendent $P_{\mathrm{III}}(\alpha, \gamma, \beta, \delta : \eta)$ and so we have as a solution of eq. (4.24)

$$Z = P_{\mathrm{III}}\Big(0, 0, \frac{\delta_0}{8}, -\frac{\delta_0}{8}; \eta\Big). \tag{4.25}$$

This equation is, however, not new; it can be transformed into solution (4.15) by making use of relations between special cases of $P_{\mathrm{V}}$ and $P_{\mathrm{III}}$.

### D. Comments on the Painlevé analysis and integrability of model

For $A \neq 0$ eq. (4.5) passes the Painlevé test for any function $g(\xi)$ and $h^2(\xi) - l(\xi)$. Indeed, we find $p = -1$ in the expansion (4.2). A resonance is obtained for $k = 1$; the resonance condition is satisfied and so the coefficient $a_1$ is a free constant (as is $x_0$).

The Painlevé test only checks whether certain necessary conditions are satisfied. If an equation of the type (4.1) does actually have the Painlevé property (as opposed to merely passing the Painlevé test), then it can be transformed into a standard form by a Möbius transformation (with variable coefficients)

$$y(\xi) = \frac{\alpha(\xi)U(\eta(\xi)) + \beta(\xi)}{\delta(\xi)U(\eta(\xi)) + \rho(\xi)}, \quad \eta = \eta(\xi), \quad \alpha\rho - \beta\delta = \pm 1. \tag{4.26}$$

Eq. (4.5) is already, to a large extend, standardized. Indeed, the coefficient of $\dot{U}^2$ has poles at $U = 0$, 1 and $\infty$. This puts the equation into Ince's class IV and the residues have the correct values. Hence we have $\alpha = \rho = 1$, $\beta = \delta = 0$ in eq. (4.26). The only remaining permitted transformation is that of the independent variable. We have shown above that eq. (4.5) can be reduced to the elliptic equation if and only if $B$ in eq. (4.8) is constant. It can be transformed into the equation for the $P_{\mathrm{V}}$ transcendent if and only if its coefficients satisfy

$$\frac{d}{d\xi}\Big(\frac{h^2 - l}{g^2}\Big) - 2\Big(\frac{h^2 - l}{g^2}\Big)g = 0. \tag{4.27}$$

In all other cases the equation (4.5) cannot be transformed into a standard form and hence it does not have Painlevé property.

The situation is exactly the same for $A = 0$. Eq. (3.10) passes the Painlevé test and is transformed into eq. (4.19) by a Möbius transformation. The coefficient of $\dot{Z}^2$ has poles at $Z = 0$ and $Z \to \infty$ with the correct residues. Hence only $Z(\xi) \to \alpha(\xi)Z\big(\eta(\xi)\big)$ is permitted. Eq. (4.19) is of Ince's type II and has the Painlevé properties if and only if eq. (4.27) is satisfied.

Thus we have shown that eq. (4.5) has the Painlevé property if and only if the coefficients satisfy $B = \text{const}$, or Eq. (4.27).

## V. EXPLICIT SOLUTIONS

We have reduced the original system (1.5), (1.6) for the function

$$W = Re^{i\psi} \tag{5.1}$$

to one of the pairs of equations { (3.9), (3.10)}, or $\{(3.14),(3.15)\}$.

Let us first look at the pair {(3.9), (3.10)}. Eq. (3.9) provides $F(\xi)$ by a quadrature, once eq. (3.10) is solved. In section IV we have further reduced eq. (3.10). Using eq. (4.4) we replace equations for $R(\xi)$ by equations for $U(\xi)$, where $U(\xi)$ must satisfy $U(\xi) \leq 0$.

As mentioned above, the algebras 1 and 2 of Table I lead to solutions of the form (4.15), i.e. the fifth Painlevé transcendent $P_{\mathrm{V}}(\xi)$, for $a = 0$, $b \neq 0$.

Algebras 1–23 lead to the elliptic function equation (4.9) for $m = 0$, $A \neq 0$, $B = \text{const}$ and to eq. (4.16) for $m = 0$, $A = 0$, $B = \text{const}$, $K \neq 0$. In both cases, elementary solutions are obtained in the case of multiple roots of the polynomial on the right hand side of the equation. Let us discuss in more detail the real nonsingular solutions of these equations. The character of the solution depends crucially on the sign of $B$ in the table ($B$ is defined in (3.5)). We have

- $B > 0$ for algebras

$$\begin{gathered} 1\ (b = 0, a \neq 0), 10\ (a^2 + b^2 \neq 0), 11\ (a \neq 0), 12\ (a \neq 0), \\ 15\ (a^2 > b^2), 16\ (b \neq 0), \\ 17\ (a = 0, b \neq 0, x^2 + y^2 - t^2 > 0), 18\ (a = 0, b \neq 0, x^2 + y^2 - t^2 > 0), \\ 19\ (a = 0, b \neq 0, x^2 + y^2 - t^2 > 0). \end{gathered} \tag{5.2}$$

- $B < 0$ for algebras

$$\begin{gathered} 2\ (b = 0, a \neq 0), 13\ (a \neq 0), 14\ (ab \neq 0), \\ 15\ (a^2 - b^2 < 0), \\ 17\ (a = 0, b \neq 0, t^2 - x^2 - y^2 > 0), 18\ (a = 0, b \neq 0, t^2 - x^2 - y^2 > 0), \\ 19\ (a = 0, b \neq 0, t^2 - x^2 - y^2 > 0). \end{gathered} \tag{5.3}$$

In all other cases with $m = 0$ we have $B = 0$

Let us first run through all elementary functions solutions, remembering that the independent variable is $\eta$ given in eq. (4.6).

Localized solutions are obtained precisely for the algebras (5.2). From eq. (4.16) (i.e. $A = 0$) we obtain a kink in $R(\xi)$ where $\xi$ and function $h(\xi)$ are read off from Table I. Two cases are to be considered.

I. $A = 0$, $L = -2$, $K < 0$, $B > 0$:

The solution is:

1.

$$R = \pm \tanh \frac{1}{2}\sqrt{\frac{B}{2}}(\eta - \eta_0), \quad F = -\int h(\eta)\, d\eta + F_0 \tag{5.4}$$

II. $A \neq 0$, $B > 4(A^2 - K) > 0$, $K < 0$:

Eq. (4.9) ($A \neq 0$) leads to solitary wave ("bump" or "well" type solutions) for the function $U(\eta)$ in the following cases:

2. $U_4 = U_3 = U_2 < U \leq U_1 < 0$

$$U = U_2 + \frac{U_1 - U_2}{1 + (U_1 - U_4)^2 A^2 (\eta - \eta_0)^2}. \tag{5.5}$$

Eq. (5.5) represents an "algebraic bump".

3. $U_4 \leq U < U_3 = U_2 = U_1 < 0$

$$U = U_1 - \frac{U_1 - U_4}{1 + (U_1 - U_4)^2 A^2 (\eta - \eta_0)^2}. \tag{5.6}$$

This is an "algebraic well".

4. $U_4 < U_3 = U_2 < U \leq U_1 < 0$

$$U = U_2 + \frac{(U_1 - U_2)(U_2 - U_4)}{(U_1 - U_4)\cosh^2 A\sqrt{(U_1 - U_2)(U_2 - U_4)}(\eta - \eta_0) - (U_1 - U_2)}. \tag{5.7}$$

Eq.(5.7) is an "exponential bump".

5. $U_4 \leq U \leq U_3 = U_2 < U_1 < 0$

$$U = U_2 - \frac{(U_1 - U_2)(U_2 - U_4)}{(U_1 - U_4)\cosh^2 A\sqrt{(U_1 - U_2)(U_2 - U_4)}(\eta - \eta_0) - (U_2 - U_2)}. \tag{5.8}$$

This is an "exponential well".

Further elementary solutions of eq. (4.9) are trigonometrically periodic.

6. $U_4 = U_3 < U_2 \leq U \leq U_1 < 0$, $K < 0$, $B > 4(A^2 - K) > 0$

$$U = U_3 + \frac{(U_1 - U_3)(U_2 - U_3)}{U_2 - U_3 + (U_1 - U_2)\cos^2 A\sqrt{(U_1 - U_3)(U_2 - U_3)}(\eta - \eta_0)} \tag{5.9}$$

This type of solution also occurs only for the algebras (5.2)

7. $U_4 \leq U \leq U_3 < U_2 = U_1$

$$U = U_1 - \frac{(U_1 - U_4)(U_1 - U_3)}{U_1 - U_3 + (U_3 - U_4)\cos^2 A\sqrt{(U_1 - U_4)(U_1 - U_3)}(\eta - \eta_0)} \tag{5.10}$$

This solution can occur in the case of algebras (5.2) for $U_1 < 0$, i.e. all roots negative (and then conditions (5.5) hold). It can also occur for $U_3 < 0 < U_2 = U_1$ and this allows us to have $B \leq 0$, Thus, solutions (5.10) can occur for all algebras (and variables $\xi$) 1–23 in Table I. Notice however, that they are periodic, rather than localized, in the variable $\eta$.

The remaining solutions are periodic and expressed in terms of Jacobi elliptic functions. We have:

8. $A = 0$, $K > 0$, $B < -8K < 0$, $U_2 \leq U \leq U_1 < 0$

$$U = \frac{U_1 U_2}{U_2 + (U_1 - U_2)\,\mathrm{sn}^2\left(\sqrt{-\frac{U_2 K}{2}}(\eta - \eta_0), k\right)}, \quad k^2 = \frac{U_1 - U_2}{(-U_2)} \tag{5.11}$$

This occurs for the algebras (5.3)

9. $A = 0$, $K < 0$, $B > -8K > 0$, $U_2 < U_1 \leq U \leq 0$

$$R = \sqrt{-U_1}\,\mathrm{sn}\sqrt{\frac{U_2 K}{2}}(\eta - \eta_0, k), \quad k^2 = \frac{U_1}{U_2} \tag{5.12}$$

The algebras concerned are those of eq. (5.2)

10. $A \neq 0$, $U_4 \leq U \leq U_3 < U_2 < U_1$

$$\begin{gathered} U = \frac{U_1(U_3 - U_4)\,\mathrm{sn}^2[\beta(\eta - \eta_0), k] + U_4(U_1 - U_3)}{(U_3 - U_4)\,\mathrm{sn}^2[\beta(\eta - \eta_0), k] + U_1 - U_3}, \\ k^2 = \frac{(U_1 - U_2)}{(U_1 - U_3)}\frac{(U_3 - U_4)}{(U_2 - U_3)} \quad \beta = A\sqrt{(U_1 - U_3)(U_2 - U_4)} \end{gathered} \tag{5.13}$$

This can occur for $U_1 < 0$, then we must have $B > 4(A^2 + (-K)) > 0$, i.e. the algebras (5.2). It can also occur for $U_3 < U_4 < 0 < U_2 < U_1$, then all of the algebras 1–23 of Table I can occur.

11. $A \neq 0$, $U_4 < U_3 < U_2 \leq U \leq U_1 < 0$

$$U = \frac{U_4(U_1 - U_2)\,\mathrm{sn}^2[\beta(\eta - \eta_0), k] + U_1(U_2 - U_4)}{(U_1 - U_2)\,\mathrm{sn}^2[\beta(\eta - \eta_0), k] + U_2 - U_4}, \tag{5.14}$$

with $k^2$ and $\beta$ as in eq. (5.13). we must have $B > 4(A^2 + (-K)) > 0$ and hence algebras (5.2).

12. $A \neq 0$, $U_4 < U < U_1$, $U_{2,3} = p \pm iq$, $q > 0$

$$\begin{gathered} U = \frac{[CU_4 - DU_1]\,\mathrm{cn}[\beta(\eta - \eta_0), k] + DU_1 + CU_4}{(C - D)\,\mathrm{cn}[\beta(\eta - \eta_0), k] + C + D}, \\ C = (U_1 - p)^2 + q^2, \quad D = (U_4 - p)^2 + q^2 \\ k^2 = \frac{(U_1 - U_4)^2 - (C - D)^2}{4CD}, \quad \beta = 2A(CD)^{1/4} \end{gathered} \tag{5.15}$$

This situation can occur for all algebras 1–23 of Table I.

13. $A = 0$, $K = 0$, $B < 0$. We obtain the solution (4.17) for algebras (5.3) with $\xi$ as given in Table I (and $\eta$ given by eq. (4.6)).

The algebras No. 24–29 of Table II correspond to variables $\xi$ such that $(\nabla\xi)^2 = 0$ and hence to first order ODEs. The solutions are readily obtained and we just list them.

$$\text{No. 24:} \qquad R = R_0, \psi = ay - bx + \frac{a^2+b^2}{2b}(x+t) + \psi_0, b \neq 0 \tag{5.16}$$

$$\text{No. 25:} \qquad R = c\sqrt{x+t} \pm \sqrt{c^2(x+t)-1}, \psi = \frac{b}{2}(t-x) - \frac{(a+by)^2}{2b(x+t)} + \psi_0 \tag{5.17}$$

$$\text{No. 26:} \qquad R = c\sqrt{x+t} \pm \sqrt{c^2(x+t)-1}, \psi = b\ln\sqrt{\frac{t^2-x^2-y^2}{x+t}} + \psi_0 \tag{5.18}$$

$$\text{No. 27:} \qquad R = R_0, \psi = \frac{b}{2}\ln(x+t) + \psi_0 \tag{5.19}$$

No. 28 and 29 provide nonconstant solutions only for $b = 0$. Then $F(\xi)$ and $R(\xi)$ are arbitrary functions of $\xi = x + t$.

All solutions presented so far are group invariant solutions in the standard sense of the words [8–10].

Let us mention that the PDEs (1.5) and (1.6) can be reduced to ODEs of the form (3.6) and (3.7), by the transformation (3.1) where $\xi$ and $\alpha$ are any functions satisfying eq. (3.3). The restriction is that $p$ $h$, $s$ and $l$ must be functions of $\xi$. Group theory generates solutions of these equations by the requirement that $F$ and $\xi$ in (3.1) be invariants of subgroups of the symmetry group. However, other solutions may exist, corresponding e.g. to so called "null variables" [21,22], to "conditional symmetries" [23,24], or simply generated by the "direct method" of Clarkson and Kruskal [25].

Let us just give some examples of such variables.

First a few words about null variables and the corresponding solutions. Consider a variable $\xi$ satisfying

$$(\nabla\xi)^2 = \Delta\xi = 0 \tag{5.20}$$

The equations for $F(\xi)$ and $R(\xi)$ reduce to eq. (3.13). As mentioned in Section III, if we have also

$$(\nabla\alpha, \nabla\xi) = (\nabla\alpha)^2 = \Delta\alpha = 0 \tag{5.21}$$

(e.g. for $\alpha = \text{const}$), then $F(\xi)$ and $R(\xi)$ are arbitrary functions. We have already encountered this situation for $\xi = x + t$, however eq. (5.20) have more general solutions [21,22], that can be written in terms of Riemann invariants.

Indeed let us put

$$\xi = H(\sigma), \quad \sigma = (\vec{a}, \vec{x}) = a_0 t - a_1 x - a_2 y, \quad (\vec{a}, \vec{a}) = 0 \tag{5.22}$$

where $\vec{a}$ is a lightlike vector, depending on $\xi$ (i.e. eq. (5.22) defines $\xi$ implicitly). It is easy to check that $\xi$ of eq. (5.22) satisfies eq. (eq5.20) for any choice of the function $H$ and lightlike vector $\vec{a}(\xi)$. The function $H(\sigma)$ can be chosen to be $H(\sigma) = \sigma$ with no loss of generality, since $F(\xi)$ and $R(\xi)$ are themselves arbitrary. Thus we can replace eq. (5.22) by

$$\xi = a_0(\xi)t - a_1(\xi)x - a_2(\xi)y, \quad \vec{a}^2 = 0 \tag{5.23}$$

Choosing $\vec{a}$ to be constant, we recover the variable $\xi = x + t$ (up to a Lorentz transformation). Other choices give different results, that become explicit if we can solve the algebraic equation (5.23). For example choose

$$\vec{a} = (1, \xi, \sqrt{1 - \xi^2}). \tag{5.24}$$

Solving eq. (5.23) for $\xi$, we obtain

$$\xi = \frac{(1+x)t \pm \sqrt{(1+x)^2 + y^2 - t^2}}{(1+x)^2 + y^2}. \tag{5.25}$$

Choosing $F(\xi)$ and $R(\xi)$ appropriately, e.g. $F(\xi)$ constant and $R(\xi)$ with compact support we obtain a localized solution (localized in the variable $\xi$).

An example of a "conditionally invariant" solution is obtained by putting

$$\psi = \psi(\xi), \quad R = R(\xi), \quad \xi = \sqrt{\frac{x^2 + y^2}{x^2 - t^2}}. \tag{5.26}$$

We have $\alpha(x, y, t) = 0$ and

$$p = \frac{\dot{g}}{g} = -\frac{1}{\xi}, \quad g = \frac{1}{\xi}, \quad h = s = l = m = 0, \quad B = 0 \tag{5.27}$$

in eq. (3.6) and (3.7). These values could hence be added to those in Table I.

## VI. COMMENTS AND CONCLUSIONS

Inserting the variables $\xi$ and $\alpha$ of Table I into the formulas of Section V we obtain a great variety of exact analytic solutions.

Some of our solutions are (possibly upto phase factors, contained in the variable $\alpha$) really solutions of the $1+1$, or $2+0$ dimensional $CP^1$ model. Thus, algebras 1, 11, 12, 16 provide solutions depending essentially only on $x$ and $t$. Similarly, algebras 2, 13, 14 provide essentially static solutions (independent of $t$). A sizable literature exists on static solutions [26–30]. Particularly interesting solutions of this type are obtained for algebra 2 when we have $\xi = \sqrt{x^2+y^2}$ and we take $b = 0$. We obtain elliptic function solutions (5.11) and (5.16) as well as the elementary solution (4.17), or more explicitly

$$W = W_0\Big(\sqrt{x^2+y^2}\Big)^n e^{i\,n\phi} \tag{6.1}$$

where $a = n$ is an integer (and $\phi$ is the azimuthal angle in the $xy$ plane). This can be identified as an "$n$-soliton solution" (or instanton) and it has finite energy [4]. Our static solutions are not new: they are to be found e.g. among those obtained by Purkait and Ray, or earlier [26–30].

The same algebra gives rise to a very different type of solution. If we take $b \neq 0$, $a = 0$ we express $R(\xi)$ in terms of the Painlevé transcendent $P_V$, as in eq. (4.15). The phase is $\psi = bt + \psi_0$ so that we have a global rotation of spins in the horizontal plane. To our knowledge, this type of solution is new.

Algebra 14 introduces a "helical" type variable $\xi$. Solution (4.17) in this case is

$$W = R_0(x^2+y^2)^{ab/2(1+a^2)} e^{-b/1+a^2 \arctan y/x} e^{i\,b\phi/a}. \tag{6.2}$$

This solution is multivalued, even for $b/a$ integer. This type of variable and solution could be pertinent in condensed matter applications, concerning e.g. critical phenomena in multilayered films.

In general, our method provides us with "local solutions", not necessarily defined for all of $\mathbb{R}^{\nVdash}$. The solutions are not necessarily single valued and they can have singularities for real

values of the variable $\xi$. Moreover, in view of the existence of the light cone it is sometimes necessary to consider spacelike and timelike regions of space-time separately, since solutions in these regions may differ. Typical exemples of this phenomenon are provided by algebras 17, 18 and 19. We list two variables $\xi$ in Table I for each of these, one valid for $t^2-x^2-y^2>0$, the other for $x^2+y^2-t^2>0$. In all cases we restrict to $a=0$, in order to have $m=0$ in the table. The simplest solutions are given by eq. (4.17) for $B<0$ and (5.4) for $B>0$.

In the case of algebra 17, the corresponding solutions are:

$$\begin{aligned} W &= R_0 e^{\varepsilon b\sqrt{t^2-x^2-y^2}/(x+t)} e^{-i\,by/(x+t)}, && t^2-x^2-y^2>0, \\ W &= \tanh\frac{b}{2}\left(\frac{\sqrt{x^2+y^2-t^2}}{x+t}-\xi_0\right) e^{-i\,by/(x+t)}, && t^2-x^2-y^2<0. \end{aligned} \tag{6.3}$$

The two solutions can be connected on the cone, however their derivatives will be discontinuous in any case.

Similarly, for algebra 18 of Table I we find the elementary solutions

$$\begin{aligned} W &= R_0 e^{b\arctan\sqrt{t^2-x^2-y^2}/y} e^{-i\,b\,\mathrm{arctanh}\,x/t}, && t^2-x^2-y^2>0, \\ W &= \tanh\frac{b}{2}\left[\mathrm{arctanh}\,\frac{\sqrt{x^2+y^2-t^2}}{y}-\xi_0\right] e^{-i\,b\,\mathrm{arctanh}\,y/t} && x^2+y^2-t^2>0. \end{aligned} \tag{6.4}$$

Finally, for algebra 19 of Table I we have

$$\begin{aligned} W &= \tanh\frac{b}{2}\left(\arctan\frac{\sqrt{x^2+y^2-t^2}}{t}-\xi_0\right) e^{-i\,b\arctan y/x}, && x^2+y^2-t^2>0, \\ W &= W_0 e^{b/2\,\mathrm{arctanh}\,\sqrt{t^2-x^2-y^2}/t} e^{-i\,b\arctan y/x}, && t^2-x^2-y^2>0. \end{aligned} \tag{6.5}$$

In many soliton-like problems in field theory we are interested in solutions which are regular in $\mathbb{R}^{\not\Vdash}$, i.e. which are valid at all times (though this condition is sometimes relaxed a bit) and which are defined for $-\infty<x<\infty$, $-\infty<y<\infty$. Among them particularly important are those whose energy is finite (as they describe localised "soliton-like" field structures).

If we restrict our attention to such field configurations we see that we should consider the energy density for our fields. As the energy density is given by

$$\rho=\frac{|W_t|^2+|W_x|^2+|W_y|^2}{[1+|W|^2]^2}, \tag{6.6}$$

we see that this gives us

$$\rho = \frac{(\xi_t^2 + \xi_x^2 + \xi_y^2)\dot{R}^2}{[1+R^2]^2} + \frac{(\psi_t^2 + \psi_x^2 + \psi_y^2)R^2}{[1+R^2]^2}, \tag{6.7}$$

where $\psi$ is given as in (3.1) and $\dot{R} = \frac{dR}{d\xi}$. We can rewrite $(\psi_t^2 + \psi_x^2 + \psi_y^2) = (\alpha_t^2 + \alpha_x^2 + \alpha_y^2) + \dot{F}^2(\xi_t^2 + \xi_x^2 + \xi_y^2) + 2\dot{F}(\xi_t\alpha_t + \xi_x\alpha_x + \xi_y\alpha_y)$ and then substitute the expression for $\dot{F}$ given by (3.9) (when $m = 0$).

To get the total energy we should integrate $\rho$ over all space

$$E = \int \rho \, dx \, dy. \tag{6.8}$$

To perform this integration, in some cases, we can replace the integration over $x$ and $y$ by an integration over $\xi$ and another conveniently chosen variable (which may have a finite or an infinite range). Thus in the cases of algebras 2, 3 and 19 of the Table I we can use $\xi$ and an angle, while in the cases of 1, 4, 11, 12, 15 and 16 $\xi$ involves only $x$ and as our variables of integration we can use $\xi$ and $y$. Clearly in these latter cases the total energy of any nontrivial solution is infinite.

The most extreme case corresponds to the algebra 10. In this case $\xi = t$, energy density is independent of $x$ and $y$ and so the total energy is infinite. In this case $\phi^3$ of (1.3) is given by $\phi^3 = (1 - R^2(\xi))/(1 + R^2(\xi))$ and is independent of $x$ and $y$, while $\phi^1$ and $\phi^2$ depend on $x$ and $y$ only through $\alpha$. Thus treating $\phi^i$ as components of a spin vector field (of unit length) we see that this solution describes very coherent movements of spins which move up and down in phase for all $x$ and $y$ and whose movements in the horizontal plane are modulated by $\alpha$ and $F(t)$.

Similar spin wave interpretations can be given to other solutions. In particular this is the case when the symmetry variable is more complicated than in the cases mentioned above. One can think of applications in condensed matter physics, the theory of nematic liquid crystals etc. and even in cosmology. In some of such systems the orientation of $\vec{\phi}$ does not matter; such cases can be described by a larger class of our solutions. At the same time we can consider $W(x, y, t)$ as a Landau-Ginzburg field which arises in many applications in

condensed matter physics (as can be checked the Landau-Ginzburg equation is very similar to the equation derived from (1.4)). Indeed, at least one version of the Landau Ginzburg equation has been treated using the group theoretical techniques applied in this article. The context was that of magnetic phenomena in external fields. [31]

However, returning to the field theory soliton-like applications, in which case the reductions 2, 3, 17, 18 and 19 are particularly relevant, we note that using an angular variable of integration makes it more likely that a given solution will describe a time evolution of a field configuration of finite energy.

Clearly it would be desirable to analyse further the physical implications of this and other solutions. We hope to be able to report on this in the near furture.

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# TABLES

## TABLE I.

| No | Algebra | $\xi$ | $\alpha$ | $p$ | $g$ | $h$ | $s$ | $l$ | $m$ | $B$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $K_1 + aZ,\ P_2 + bZ$ | $\sqrt{(t^2 - x^2)}$ | $by - a \operatorname{arctanh} \frac{x}{t}$ | $-\frac{1}{\xi}$ | $\frac{1}{\xi}$ | 0 | 0 | $-(b^2 + \frac{a^2}{\xi^2})$ | 0 | $2(b^2\xi^2 + a^2)$ |
| 2 | $P_0 + aZ,\ L + bZ$ | $\sqrt{x^2 + y^2}$ | $bt + a \arctan \frac{y}{x}$ | $-\frac{1}{\xi}$ | $\frac{1}{\xi}$ | 0 | 0 | $-(b^2 - \frac{a^2}{\xi^2})$ | 0 | $2(b^2\xi^2 - a^2)$ |
| 3 | $K_1 + aZ,\ K_2 + L$ | $\sqrt{t^2 - x^2 - y^2}$ | $-a \ln \lvert x + t \rvert$ | $\frac{-2}{\xi}$ | $\frac{1}{\xi^2}$ | 0 | 0 | 0 | 0 | 0 |
| 4 | $K_2 + aZ,\ P_0 - P_2$ | $x$ | $-a \ln \lvert y + t \rvert$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 5 | $D + aZ,\ P_0 - P_1$ | $\frac{t+x}{y}$ | $a \ln \lvert t + x \rvert$ | $\frac{-2}{\xi}$ | $\frac{1}{\xi^2}$ | 0 | 0 | 0 | 0 | 0 |
| 6 | $K_1 + \epsilon P_2 + aZ,$ $P_0 - P_1$ | $y\epsilon + \operatorname{arc\,th} \frac{x}{t} + \frac{1}{2} \ln(t^2 - x^2)$ | $a\epsilon y$ | 0 | 1 | $a$ | 0 | $a^2$ | 0 | 0 |
| 7 | $D + \epsilon(K_2 + L) + aZ,$ $P_0 - P_1$ | $\frac{2y}{t+x} + 2\epsilon \ln \lvert \frac{t+x}{2} \rvert$ | $-a \ln \lvert \frac{t+x}{2} \rvert$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 8 | $D + aK_1 + bZ,$ $P_0 - P_1$ | $\frac{x+t}{2} y^{(a-1)},\ a \neq 1,$ | $\frac{1}{2} b \ln(y)$ | $\frac{2-a}{a-1} \frac{1}{\xi}$ | $\xi^{(2-a)/(a-1)}$ | $\frac{b}{2(a-1)} \frac{1}{\xi}$ | $\frac{-b}{2(a-1)^2} \frac{1}{\xi^2}$ | $\frac{b^2}{4(a-1)^2} \frac{1}{\xi^2}$ | 0 | 0 |
| 9 | $D + K_1 + \epsilon(P_0 + P_1) + aZ,$ $P_0 - P_1$ | $e^{x+t} y^{-2\epsilon}, \quad \varepsilon = \pm 1$ | $\frac{a\epsilon}{2}(x + t)$ | $-\frac{1+2\epsilon}{2\epsilon} \frac{1}{\xi}$ | $\xi^{-(1+2\epsilon)/2\epsilon}$ | 0 | 0 | 0 | 0 | 0 |
| 10 | $P_1 + aZ;\ P_2 + bZ$ | $t$ | $by + ax$ | 0 | 1 | 0 | 0 | $-(b^2 + a^2)$ | 0 | $2(b^2 + a^2)$ |
| 11 | $D + aZ,\ P_2$ | $\frac{x}{t}$ | $a \ln t$ | $-\frac{2\xi}{\xi^2-1}$ | $\frac{1}{\xi^2-1}$ | $-\frac{a\xi}{\xi^2-1}$ | $-\frac{a}{\xi^2-1}$ | $\frac{a^2}{\xi^2-1}$ | 0 | $2a^2$ |
| 12 | $D + K_1 + \epsilon(P_0 + P_1) + aZ,\ P_2$ | $t + x - \epsilon \ln \lvert t - x \rvert$ | $\frac{a}{2} \ln \lvert t - x \rvert$ | 0 | 1 | $-\frac{\epsilon a}{4}$ | 0 | 0 | 0 | $\frac{a^2}{8}$ |
| 13 | $D + aZ,\ P_0$ | $\arctan \frac{y}{x}$ | $a \ln \sqrt{x^2 + y^2}$ | 0 | 1 | 0 | 0 | $a^2$ | 0 | $-2a^2$ |
| 14 | $D + aL + bZ,\ P_0$ | $\frac{1}{2} \ln(x^2 + y^2) - \frac{1}{a} \arctan \frac{y}{x}$ | $\frac{b}{a} \arctan \frac{y}{x}$ | 0 | 1 | $-\frac{b}{1+a^2}$ | 0 | $\frac{b^2}{1+a^2}$ | 0 | $-2\frac{b^2 a^2}{(1+a^2)^2}$ |
| 15 | $P_2 + aZ,\ P_0 + bZ$ | $x$ | $by + at$ | 0 | 1 | 0 | 0 | $(b^2 - a^2)$ | 0 | $2(a^2 - b^2)$ |
| 16 | $D + aK_1 + bZ,\ P_2,$ | $\frac{(t+x)^{a+1}}{(t-x)^{1-a}}$ $a \geq 0,\ a \neq 1,$ | $\frac{b}{1+a} \ln \lvert t - x \rvert$ | $-\frac{1}{\xi}$ | $\frac{1}{\xi}$ | $\frac{b}{2\xi(a^2-1)}$ | 0 | 0 | 0 | $\frac{b^2}{2(a^2-1)^2}$ |
| 17 | $D + aZ,\ K_2 + L + bZ$ | $\frac{(t^2-x^2-y^2)^{1/2}}{x+t}$ | $-b\frac{y}{t+x} + a \ln \lvert t + x \rvert$ | 0 | 1 | $\frac{-a}{\xi}$ | 0 | $b^2$ | $\frac{a}{\xi^2}$ | $2\left(\frac{a^2}{\xi^2} - b^2\right)$ |
| | | $\frac{(x^2+y^2-t^2)^{1/2}}{x+t}$ | $-b\frac{y}{t+x} + a \ln \lvert t + x \rvert$ | 0 | 1 | $\frac{-a}{\xi}$ | 0 | $-b^2$ | $\frac{a}{\xi^2}$ | $2\left(\frac{a^2}{\xi^2} + b^2\right)$ |
| 18 | $D + aZ,\ K_1 + bZ$ | $\arctan \frac{(t^2-x^2-y^2)^{1/2}}{y}$ | $-b \operatorname{arctanh} \frac{x}{t} + \frac{a}{2} \lvert t^2 - x^2 \rvert$ | 0 | 1 | $-\frac{a}{\tan \xi}$ | 0 | $b^2 - a^2$ | $-\frac{a}{\sin^2 \xi}$ | $2\left(\frac{a^2}{\sin^2 \xi} + b^2\right)$ |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\operatorname{arctanh}\frac{(x^2+y^2-t^2)^{1/2}}{y}$ | $-b\operatorname{arctanh}\frac{x}{t}+\frac{a}{2}\lvert x^2-t^2\rvert$ | $0$ | $1$ | $-\frac{a}{\tanh\xi}$ | $0$ | $a^2-b^2$ | $-\frac{a}{\sinh^2\xi}$ | $2\left(\frac{a^2}{\sinh^2\xi}-b^2\right)$ |
| 19 | $D+aZ,\ L+bZ$ | $\arctan\frac{(x^2+y^2-t^2)^{1/2}}{t}$ | $-b\arctan\frac{y}{x}+\frac{a}{2}\ln\sqrt{x^2+y^2}$ | $0$ | $1$ | $-\frac{a}{\tan\xi}$ | $0$ | $-a^2-b^2$ | $\frac{a}{\sin^2\xi}$ | $2\left(\frac{a^2}{\sin^2\xi}+b^2\right)$ |
| | | $\operatorname{arctanh}\frac{(t^2-x^2-y^2)^{1/2}}{t}$ | $-b\arctan\frac{y}{x}+\frac{a}{2}\ln\sqrt{x^2+y^2}$ | $0$ | $1$ | $-\frac{a}{\tanh\xi}$ | $0$ | $a^2+b^2$ | $\frac{a}{\sinh^2\xi}$ | $2\left(\frac{a^2}{\sinh^2\xi}-b^2\right)$ |
| 20 | $D-K_1+aZ,$ $P_0-P_1+bZ$ | $\frac{\sqrt{x+t}}{y}$ | $at-\frac{a}{2}(x+t)+\frac{b}{2}\ln\lvert x+t\rvert$ | $-\frac{2}{\xi}$ | $\frac{1}{\xi^2}$ | $-\frac{a}{2\xi^3}$ | $0$ | $-\frac{ab}{\xi^4}$ | $\frac{a}{2\xi^4}$ | $2(\frac{a^2}{4\xi^2}+ab)$ |
| 21 | $D+aK_1+bZ,$ $K_2+L$ | $\frac{1}{t^2-x^2-y^2}(t+x)^{2/(1-a)},$ $a\neq\pm1,$ | $\frac{b}{1-a}\ln(x+t)$ | $\frac{1}{2}\frac{3+a}{a+1}\frac{1}{\xi}$ | $\xi^{-(3+a)/(2a+2)}$ | $\frac{b}{a+1}\frac{1}{2\xi}$ | $0$ | $0$ | $\frac{b(1-a)}{4(a+1)^2}\frac{1}{\xi^2}$ | $\frac{b^2}{2(a+1)^2}\xi^{(1-a)/(1+a)}$ |
| 22 | $D-K_1+\epsilon(P_0-P_1)+aZ,$ $K_2+L$ | $\frac{x^2+y^2-t^2}{2(x+t)}-\frac{\epsilon}{2}\ln\lvert x+y\rvert,$ $\varepsilon=\pm1$ | $\frac{a}{2}\ln\lvert x+t\rvert$ | $\epsilon$ | $e^{\epsilon\xi}$ | $-\frac{\epsilon a}{2}$ | $0$ | $0$ | $\frac{a}{2}$ | $\frac{a^2e^{-2\epsilon\xi}}{2}$ |
| 23 | $D+\frac{1}{2}K_1+aZ,$ $K_2+L+\epsilon(P_0+P_1)$ | $\frac{6(t-x)+(t+x)^3+6\epsilon y(t+x)}{[(t+x)^2+4\epsilon y]^{3/2}}$ | $a\ln\lvert(t+x)^2+4\epsilon y\rvert$ | $\frac{5}{3}\frac{-\xi}{1-\xi^2}$ | $\frac{1}{1-\xi^2}^{5/6}$ | $\frac{2}{3}\frac{a\xi}{1-\xi^2}$ | $\frac{4}{9}\frac{a}{1-\xi^2}$ | $-\frac{4}{9}\frac{a^2}{1-\xi^2}$ | $\frac{2a}{9(1-\xi^2)^2}$ | $\frac{8a^2}{9}(1-\xi^2)^{-1/3}$ |

TABLE II.

| No | Algebra | $\xi$ | $\alpha$ | $\Delta\xi$ | $(\nabla\xi)^2$ | $\Delta\alpha$ | $(\nabla\alpha,\nabla\xi)$ | $(\nabla\alpha)^2$ |
|---|---|---|---|---|---|---|---|---|
| 24 | $P_2+aZ,\ P_0-P_1+bZ$ | $x+t$ | $-bx+ay$ | 0 | 0 | 0 | $b$ | $-b^2-a^2$ |
| 25 | $K_2+L+aZ,\ P_0-P_1+bZ$ | $x+t$ | $bt-\frac{2ay+by^2}{2(x+t)}$ | 0 | 0 | $\frac{b}{x+t}$ | $b$ | $b^2-\frac{a^2}{\xi^2}$ |
| 26 | $D+K_1+bZ,\ K_2+L$ | $x+t$ | $\frac{b}{2}\ln(t^2-x^2-y^2)$ | 0 | 0 | $\frac{b}{t^2-x^2-y^2}$ | $\frac{b(t+x)}{t^2-x^2-y^2}$ | $\frac{b^2}{t^2-x^2-y^2}$ |
| 27 | $D-K_1+bZ,\ K_2+L$ | $\frac{x+t}{t^2-x^2-y^2}$ | $\frac{b}{2}\ln(t+x)$ | $-2\frac{(x+t)}{(t^2-x^2-y^2)^2}$ | 0 | 0 | $-b\frac{(t+x)}{(t^2-x^2-y^2)^2}$ | 0 |
| 28 | $D+K_1+bZ,\ P_2$ | $x+t$ | $\frac{b}{2}\ln(t-x)$ | 0 | 0 | 0 | $\frac{b}{t-x}$ | 0 |
| 29 | $D+K_1+bZ,\ P_0-P_1$ | $x+t$ | $\alpha=b\ln y$ | 0 | 0 | $\frac{b}{y^2}$ | 0 | $-\frac{b^2}{y^2}$ |